In addition to the equations of motion we have conservation of mass

$$\nabla \cdot (\rho_s \mathbf{v}_s + \rho_n \mathbf{v}_n) + \frac{\partial \rho}{\partial t} = 0,$$

and in the counterflow experiments to be discussed the total momentum also vanishes

$$\rho_s \bar{\mathbf{v}}_s + \rho_n \bar{\mathbf{v}}_n = 0, \tag{3}$$

where the bars denote averaging across the slit width.

In steady state flow local accelerations vanish so on the left side of (1) and (2) only the second order terms remain. Adding (1) and (2) we get

$$\begin{aligned} \rho_s (\mathbf{v}_s \cdot \nabla) \mathbf{v}_s + \rho_n (\mathbf{v}_n \cdot \nabla) \mathbf{v}_n = -\nabla P - \eta_n \nabla \times (\nabla \times \mathbf{v}_n) \\ + (2\eta_n + \eta') \nabla (\nabla \cdot \mathbf{v}_n). \end{aligned} \tag{4}$$

The heat current density $\mathbf{q}$ (watts/cm^2) is carried by the normal fluid such that

$$\mathbf{q} = \rho s T \mathbf{v}_n = \mathbf{v}_n \beta^{-1} \tag{5}$$

where $\beta \equiv (\rho s T)^{-1}$. Since heat is assumed to be conserved, $\nabla \cdot \mathbf{q} = 0$ and

$$\nabla \cdot \mathbf{v}_n = \beta \nabla \cdot \mathbf{q} + \mathbf{q} \cdot \nabla \beta = \mathbf{q} \cdot \nabla \beta. \tag{6}$$

Using the vector identity

$$\nabla \times (\nabla \times \mathbf{v}) = \nabla (\nabla \cdot \mathbf{v}) - \nabla^2 \mathbf{v}$$

the terms in (4) involving viscosity may be simplified to give

$$\nabla P = \eta_n \nabla^2 (\beta \mathbf{q}) + (\eta_n + \eta') \nabla (\mathbf{q} \cdot \nabla \beta) - \rho_s (\mathbf{v}_s \cdot \nabla) \mathbf{v}_s - \rho_n (\mathbf{v}_n \cdot \nabla) \mathbf{v}_n . \tag{7}$$

To solve this equation we now make some assumptions which will later be shown to be valid for long narrow slits. Take the z axis along the length of the slit, and the x axis across the slit. Unit vectors in these directions are $\mathbf{e}_z$ and $\mathbf{e}_x$. We assume that

$$\mathbf{q} = q(x) \mathbf{e}_z \quad \text{and} \quad T = T(z) \tag{8}$$

thereby implying that $\beta = \beta(z)$ and $\eta_n = \eta_n(z)$. We further assume that the second order terms on the right of (7) are small compared to the other terms. Using these assumptions (7) may be separated into x and z components to give[2]

$$\frac{\partial P}{\partial z} = \eta_n \beta \frac{d^2 \mathbf{q}}{dx^2} + (2\eta_n + \eta') \mathbf{q} \frac{d^2 \beta}{dz^2} \tag{9}$$

[2] Most of the following equations in which the heat current density appears are not vector equations. Nevertheless, for convenience we continue to use the boldface notation $\mathbf{q}$ for this quantity.